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AN ANALYSIS OF OBSERVED-SYSTEM SENSITIVITY TO PLANT PARAMETER V--ETC(U)

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THESIS

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Capt USAF

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AN ANALYSIS OF OBSERVED-SYSTEM SENSITIVITY TO PLANT PARAMETER VARIATIONS

THESIS

Presented to the Faculty of the School of Engineering
of the Air Force Institute of Technology
Air University
in Partial Fulfillment of the
Requirements for the Degree of
Master of Science

by
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Graduate Guidance and Control

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Preface

This topic was selected for study as a result of my fascination with observers. Although often overlooked, or replaced by a Kalman filter, the observer is a relatively simple solution to many sometimes-complex problems. In this report, a simple example plant was selected for study because I felt it would provide as much insight to the problem as would a more comprehensive model. Also, many practical control systems are, in reality, no more complex that the two-state model investigated. All of the cases worked were designed to reflect practical problems that could possibly be encountered in a real-life situation. I sincerely hope that these results will be a useful contribution to the control world.

A special note of appreciation is in order for four outstanding persons who made this project a little more bearable. My lovely wife, Peggy, stood by me, offering encouragement when everything seemed to go wrong and rejoiced with me when obstacles were overcome. My initial thesis advisor, Major Richard M. Potter, helped immeasurably in setting up a solid research effort and getting me started on the problem. Captain J. Gary Reid took over as advisor well into the study, picking up loose ends and molding my efforts into the final stretch. His patient understanding and guidance in generating the computer

program used for solving example cases were of immeasurable assistance in accomplishing the overall task. Finally a note of appreciation is necessary for my typist, Phyllis Reynolds, for her outstanding work and professional touch that spared me the normal hassles involved in typing a thesis. To these four I am especially indebted.

Dennis L. Hamme

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Abstract

State-variable feedback is a modern control theory technique that is employed in system design to place closed-loop poles to achieve desired performance characteristics. Two problems associated with state-variable feedback are physical inaccessibility to plant states and output sensitivity to plant parameter variations. In this report an observer is employed to reconstruct all plant states. The plant-observer system is investigated with respect to sensitivity aspects of the following three areas: state-variable representation, extent of pole placement, and observer dynamics design. A comparison between physical, phase, and Jordan canonical variables indicated physical and phase variable representations yield identical sensitivity functions; Jordan canonical variables result in greater sensitivities than physical and phase variables. Placing plant poles close together creates a high system sensitivity to plant parameter variations; separation of poles reduces sensitivity. As a nominal requirement for low system sensitivity, observer poles should be placed to the left of dominant plant poles; further reductions in sensitivity are achieved by placing observer poles to the left of all plant poles.

AN ANALYSIS OF OBSERVED-SYSTEM SENSITIVITY TO PLANT PARAMETER VARIATIONS

I. Introduction

Conventional control theory introduces the use of compensators placed either in cascade with or feedback around basic plant elements to improve closed-loop performance by modifying the pole-zero pattern. D'Azzo and Houpis demonstrate that the feedback of system states through appropriate gains can rearrange the poles to achieve almost any desired closed-loop performance (Ref 1:410-413). This state-variable feedback technique is accomplished by constructing the closed-loop transfer function, or control ratio, in terms of feedback coefficient, k_i, for each i-th system state. Using the performance specifications as a guide, a desired control ratio is synthesized to achieve the proper response. The final step is to equate the two control ratios to determine the required k_i. There are two major problems associated with state-variable feedback which must be considered: all system states may not be physically available for feeding back to the input and the resulting system performance may be quite sensitive to parameter variations within the plant.

The problem of inaccessible states has been investigated thoroughly and several solutions proposed. Minor-loop

feedback of accessible states can be employed to synthesize inaccessible states (Ref 2:517-529). This approach, however, requires exact knowledge of the plant parameters for faithful duplication of the states. A parameter variation or inaccuracy in determining plant parameters can render this procedure impractical. A second solution is to design an outer-loop feedback compensator through block diagram manipulation (Ref 1:441-444). This approach requires utilization of differentiator devices, which are inherently noisy, and thus are not suitable for practical applications. A third approach is to derive the state vector of a plant through modelling techniques, such as with a Kalman filter or an observer.

Luenberger developed the mathematical basis for a linear system that, when driven by the inputs and outputs of a second system, could reconstruct the entire state vector of the second system (Ref 3:74-80). Termed an observer, this system actually constructs an invertible, linear transformation of the state vector of the observed system and has no ill effects upon the control ratio other than adding its own poles. These are selected by the designer as a prudent choice of observer dynamics. To better understand this, the design procedure is presented below. Given a linear, continuous, time-invariant system of the form

$$\underline{\dot{x}}(t) = A\underline{x}(t) + B\underline{u}(t) \qquad (1)$$

and $$\underline{y}(t) = C\underline{x}(t) \tag{2}$$

where $\underline{x}(t)$ is an nxl state vector

$\underline{u}(t)$ is an mxl input vector

$\underline{y}(t)$ is an rxl output vector

A is an nxn transition, or dynamics, matrix

B is an nxm distribution, or control, matrix

C is an rxn output matrix

an observer that will reconstruct a linear transformation of the state vector, $\underline{z}(t)=T\underline{x}(t)$, is given by

$$\dot{\underline{z}}(t) = D\underline{z}(t) + H\underline{x}(t) + F\underline{u}(t) \tag{3}$$

where $$TA-DT=H \tag{4}$$

and $$F=TB \tag{5}$$

The matrix H is selected to force the observer dynamics matrix, D, to have some desired eigenvalues. Equations (4) and (5) are then solved to complete Eq (3). The matrix T must be invertible (nonsingular) to ensure the recovery of the reconstructed state vector $\hat{\underline{x}}(t)=T^{-1}\underline{z}(t)$. There is also a stipulation that the initial conditions of the system and observer state vectors must be equal. Luenberger proved that if the above conditions were satisfied, the observer would exactly reproduce the selected transformation of the system state vector (Ref 3:75).

In practical applications the system output vector, rather than the state vector, drives the observer. Then

Eq (3) becomes

$$\underline{\dot{z}}(t) = D\underline{z}(t) + V\underline{y}(t) + F\underline{u}(t) \qquad (6)$$

or, by Eq (2),

$$\underline{\dot{z}}(t) = D\underline{z}(t) + VC\underline{x}(t) + F\underline{u}(t) \qquad (7)$$

wherein Eq (4) becomes

$$TA-DT=VC \qquad (8)$$

This simplifies the calculation of D since only nr elements of V must be chosen instead of n^2 elements of H.

An observer is employed to provide the inaccessible states for feedback to the input to obtain the benefits of state-variable feedback. In fact, the entire reconstructed state vector can be utilized for feedback to minimize instrumentation costs by not having to measure the accessible system states. As mentioned earlier, only the observer poles are added to the closed-loop transfer function and they may be placed where desired by the designer; the plant poles are shifted independently by choice of feedback coefficients.

One significant question arises concerning the use of observers to enhance state-variable feedback: how is the control ratio affected by plant parameter variations? In general, given any black box with output y(t), the change in output due to a change in an internal parameter is expressed by a first-order Taylor series approximation as

$$y(t;p_o+\Delta p) \simeq y(t;p_o) + \left.\frac{\partial y(t;p)}{\partial p}\right|_{p_o} \Delta p \qquad (9)$$

where p_o is the nominal parameter value and Δp is the variation. If the system is linear, the steady-state output can be considered with no loss of generality. Rewriting Eq (9) as

$$y_{ss}(t;p_o+\Delta p) - y_{ss}(t;p_o) \approx \left.\frac{\partial y_{ss}(t;p)}{\partial p}\right|_{p_o} \Delta p \qquad (10)$$

it is seen that the change in output is approximately equal to the "sensitivity" of $y(t)$ to p scaled by the variation in p. Since the sign and magnitude of Δp are not known, the area of concern lies in the magnitude of the sensitivity function. For greater understanding of how the system is affected by Δp, a frequency response of the sensitivity function would reveal any particular frequencies of operation for which the system would be especially susceptible to Δp.

Consider, for example, the linear regulator in Fig. 1. The output of the plant is fed back to nullify disturbance inputs, $d(t)$. If the output is particularly sensitive to a certain band of frequencies within the operational bandwidth, the feedback

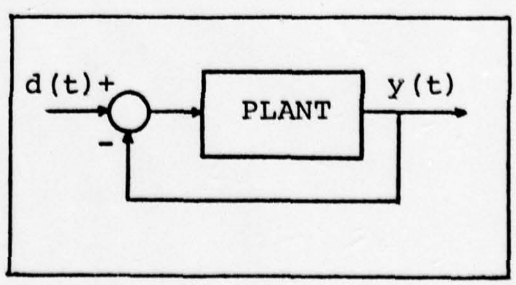


Figure 1. Linear Regulator

signal could itself introduce disturbance inputs for some parameter variation within the plant.

Another example is in the control system in Fig. 2.

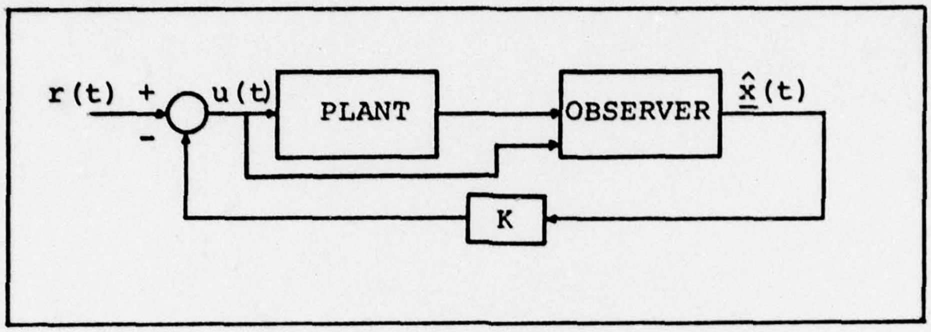


Figure 2. Feedback Control System

Utilizing state-variable feedback, an input r(t) causes a desired steady-state output y(t). The feedback coefficient matrix, K, is selected to provide a particular response and the observer constructs the required state vector. Under a parameter variation the plant output may vary, causing errors in the observer which, in turn, generates the wrong feedback signal that creates more errors--and the cycle continues. Or, quite possibly, the plant parameters are not known exactly and the observer design is based upon "best guess" parameters. The reconstructed state vector will contain an error such that the feedback signal does not provide the desired system response. In both cases the sensitivity function provides insight into the severity of output deviations due to parameter variations. A frequency response of the sensitivity function can assist in determining whether or not the system can perform

satisfactorily within its operating bandwidth.

The objective of this thesis is to compare the sensitivity functions for direct state feedback and observer-implemented feedback systems to determine if the employment of observers is acceptable in light of plant parameter variations.

II. Development of the Study

There are three major areas to be investigated for their effects upon system sensitivity to parameter variations: state-variable representation, pole placement, and observer design. These topics will be studied both individually and in combinations to test for possible interdependence.

It has been demonstrated that the choice of state-variable representation can have a rather drastic effect upon eigenvalue sensitivity. A small change in a plant parameter can cause the eigenvalues to vary by many orders of magnitude or very little at all (Refs 4:85-93; 5:263-266). The concept of eigenvalue sensitivity centers around simulation of a linear, time-invariant system on a computer. A high sensitivity indicates an eigenvalue of the system may vary greatly from its nominal value in the presence of elemental variations or inaccuracies in modelling. The resulting errors in system dynamics detract from the benefits of modelling. The correlation between eigenvalue sensitivity and closed-loop system sensitivity to parameter variations is not well-defined; therefore, a comparison will be made between three common types of state variables to determine whether the type of state-variable representation affects system sensitivity. Physical variables are selected because they depict actual,

measurable quantities that can be utilized for feedback. Phase canonical, or simply phase, and Jordan canonical variables are purely mathematical quantities that have practical applications in control system analysis and design. Singer shows that these two forms can have radically different eigenvalue sensitivities (Ref 4:88) and, thus, are logical candidates for study.

The second area of interest is the pole placement that results from state-variable feedback. It seems possible that the distance poles are shifted and/or the pattern desired (e.g., double roots) may affect system sensitivity because of the range of possible feedback gains. An initial guess is that larger feedback gains would tend to decrease system sensitivity due to increased feedback effects, except that errors in the reconstructed state vector would be amplified. The converse seems logical for small feedback gains. This facet of the problem will be studied to determine if, indeed, there exists any correlation between pole placement and sensitivity.

There have been many articles published on observer theory (see, for example Refs 7, 8, and 9) but none of them specifically address general guidance in observer pole selection. Luenberger states only that the observer dynamics should have a quicker response than the dynamics of the plant it observes (Ref 6:190). An obvious solution is to place the observer poles far into the left half of

the s plane, but this introduces differentiator noise as the poles become very large negatively. This question will be investigated with the goal of establishing some guideline for selecting observer poles to minimize system sensitivity to parameter variations.

The question of what constitutes a sensitivity function has no standard answer. Anderson (Ref 10:117-124) considers the return difference matrix, $I+G(jw)H(jw)$, where $G(jw)H(jw)$ is the open-loop transfer function, as the determining factor. Since the closed-loop transfer function is $G(jw)/[I+G(jw)H(jw)]$, a system obviously is insensitive to a parameter variation in the plant, $G(jw)$, if the return difference is "large." On the other hand, D'Azzo and Houpis define a sensitivity function to be the ratio of derivatives of the natural logarithms of the control ratio and the variable parameter. Their algebraically-reduced sensitivity function is

$$S_p^M(jw) = \frac{p}{M}\frac{\partial M}{\partial p} \tag{11}$$

where S is the sensitivity of the control ratio, M, with respect to parameter p. Equation (11) is a normalized form of the more general sensitivity function, $\partial y/\partial p$, (Ref 13:1) which simply expresses the change in output due to a parameter variation. This definition follows from Eq (10) and will be used as the standard definition of sensitivity throughout this report.

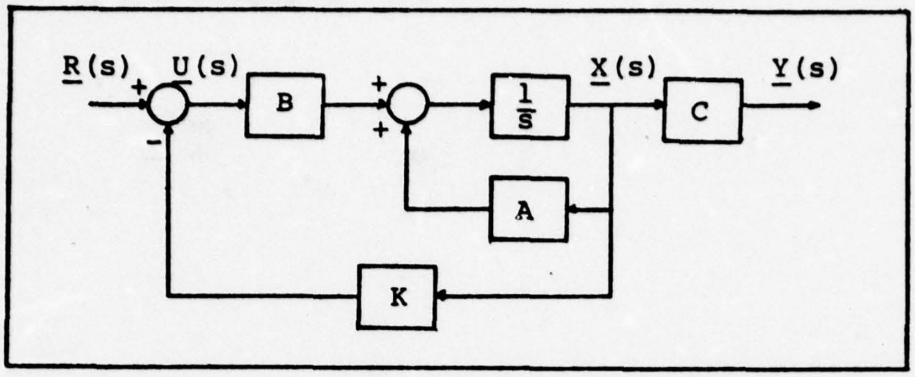


Figure 3. State-variable Feedback

Figure 3 depicts the application of state-variable feedback to a linear system that can be modelled by a first-order differential equation. The Laplace transform of the describing differential equation for the system is

$$s\underline{X}(s) = A\underline{X}(s) + B\underline{U}(s) \tag{12}$$

with output $$\underline{Y}(s) = C\underline{X}(s) \tag{13}$$

and input $$\underline{U}(s) = -K\underline{X}(s) + \underline{R}(s) \tag{14}$$

Substituting Eq (14) into Eq (12) and rearranging terms yields

$$s\underline{X}(s) = (A-BK)\underline{X}(s) + B\underline{R}(s) \tag{15}$$

Taking the partial derivative of Eqs (15) and (13) with respect to parameter p yields, respectively,

$$\frac{\partial s\underline{X}(s)}{\partial p} = \frac{\partial(A-BK)}{\partial p}\,\underline{X}(s) + (A-BK)\,\frac{\partial \underline{X}(s)}{\partial p} + \frac{\partial B}{\partial p}\,\underline{R}(s) \tag{16}$$

$$\frac{\partial \underline{Y}(s)}{\partial p} = \frac{\partial C}{\partial p}\,\underline{X}(s) + C\,\frac{\partial \underline{X}(s)}{\partial p} \tag{17}$$

It is noted that the input $\underline{R}(s)$ is not a function of the variable parameter. The partial derivatives are evaluated at the nominal parameter value. Eqs (15) and (16) may be combined to form an augmented set of matrix equations, or

$$\left[\begin{array}{c} s\underline{X}(s) \\ \hline \dfrac{\partial s\underline{X}(s)}{\partial p} \end{array}\right] = \left[\begin{array}{c|c} A-BK & 0 \\ \hline \dfrac{\partial (A-BK)}{\partial p} & A-BK \end{array}\right] \left[\begin{array}{c} \underline{X}(s) \\ \hline \dfrac{\partial \underline{X}(s)}{\partial p} \end{array}\right] + \left[\begin{array}{c} B \\ \hline \dfrac{\partial B}{\partial p} \end{array}\right] \underline{R}(s) \tag{18}$$

$$\frac{\partial \underline{Y}(s)}{\partial p} = \left[\begin{array}{c|c} \dfrac{\partial C}{\partial p} & C \end{array}\right] \left[\begin{array}{c} \underline{X}(s) \\ \hline \dfrac{\partial \underline{X}(s)}{\partial p} \end{array}\right] \tag{19}$$

These two equations have the same form as Eqs (12) and (13). For simplicity, the 2n x 2n augmented "A" matrix of Eq (18) is labelled A'. Then, solving Eq (18) for the augmented "$\underline{X}(s)$" vector and substituting it into Eq (19) yields the sensitivity transfer function

$$\frac{\partial Y(s)}{\partial p} = \left[\begin{array}{c|c} \dfrac{\partial C}{\partial p} & C \end{array}\right] \left[sI-A' \right]^{-1} \left[\begin{array}{c} B \\ \hline \dfrac{\partial B}{\partial p} \end{array}\right] \underline{R}(s) \tag{20}$$

In Fig. 4 an observer has been added to the basic system of Fig. 3 to provide the reconstructed state vector, $\hat{\underline{X}}(s)$. An acceptable choice for T is the identity matrix so that the state vector of the observer is, in steady state, the state vector of the plant it is observing.

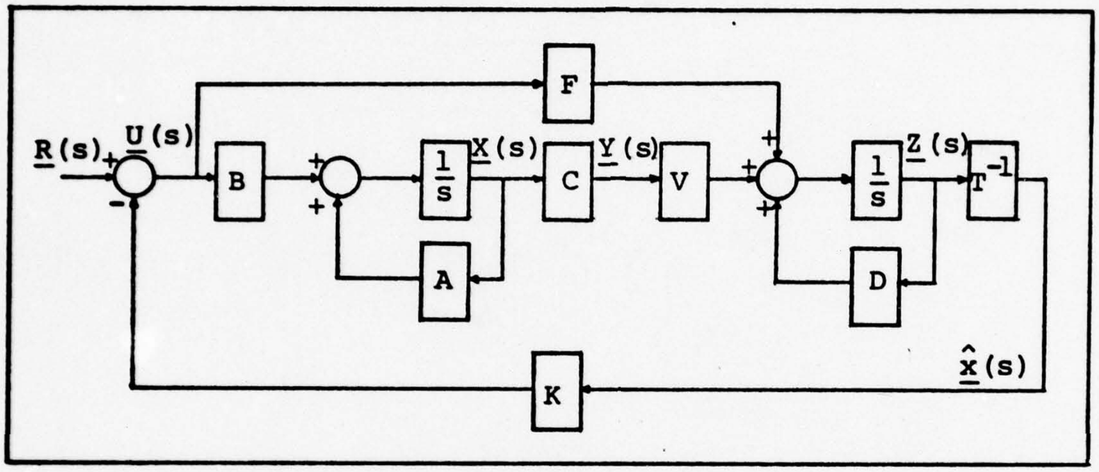


Figure 4. Use of Observer in State-variable Feedback

Choosing T=I also reduces the mathematics involved since F=B and Eq (8) becomes

$$A-D=VC \tag{21}$$

In the frequency domain, with the substitution $\underline{U}(s)= -K\underline{Z}(s)+\underline{R}(s)$, Eq (7) can be expressed as

$$s\underline{Z}(s) = (D-BK)\underline{Z}(s) + VC\underline{X}(s) + B\underline{R}(s) \tag{22}$$

Considering that the matrices D and V are not functions of the plant parameter p, since they are computed at the nominal value of p, the partial derivative of Eq (22) with

respect to p is

$$\frac{\partial s\underline{Z}(s)}{\partial p} = -\frac{\partial B}{\partial p}K\underline{Z}(s) + (D-BK)\frac{\partial \underline{Z}(s)}{\partial p} + V\frac{\partial C}{\partial p}\underline{X}(s)$$

$$+ VC\frac{\partial \underline{X}(s)}{\partial p} + \frac{\partial B}{\partial p}\underline{R}(s) \qquad (23)$$

As in the state-variable feedback case it is desired to construct an augmented matrix equation as a prelude to the sensitivity transfer function for the system in Fig. 4. Combining Eqs (12), (22), (16), and (23) gives

$$\begin{bmatrix} s\underline{X}(s) \\ s\underline{Z}(s) \\ \hline \dfrac{\partial s\underline{X}(s)}{\partial p} \\ \dfrac{\partial s\underline{Z}(s)}{\partial p} \end{bmatrix} = \left[\begin{array}{cc|cc} A & -BK & 0 & 0 \\ VC & D-BK & 0 & 0 \\ \hline \dfrac{\partial A}{\partial p} & -\dfrac{\partial B}{\partial p}K & A & -BK \\ V\dfrac{\partial C}{\partial p} & -\dfrac{\partial B}{\partial p}K & VC & D-BK \end{array}\right] \begin{bmatrix} \underline{X}(s) \\ \underline{Z}(s) \\ \hline \dfrac{\partial \underline{X}(s)}{\partial p} \\ \dfrac{\partial \underline{Z}(s)}{\partial p} \end{bmatrix} + \begin{bmatrix} B \\ B \\ \hline \dfrac{\partial B}{\partial p} \\ \dfrac{\partial B}{\partial p} \end{bmatrix} \underline{R}(s) \qquad (24)$$

As before, Eq (24) can be solved for the augmented state vector, which is substituted into Eq (17) to obtain the sensitivity transfer function. Defining the partitioned 4n x 4n "A" matrix of Eq (24) as A", the final result is

$$\frac{\partial \underline{Y}(s)}{\partial p} = \left[\begin{array}{cc|cc} \dfrac{\partial C}{\partial p} & 0 & C & 0 \end{array}\right] \left[sI - A''\right]^{-1} \begin{bmatrix} B \\ B \\ \hline \dfrac{\partial B}{\partial p} \\ \dfrac{\partial B}{\partial p} \end{bmatrix} \underline{R}(s) \qquad (25)$$

where each zero element in the row matrix above is itself an rxn matrix of zeros to maintain conformability.

This concludes the theoretical development required for the problem. Since it is obviously not practical to analytically compare Eq (20) with Eq (25) to determine if one system is more sensitive to a parameter variation than the other, the next chapter will develop numerous examples to investigate this question.

III. Numerical Examples

In order to keep the required computations to a convenient level a two-state plant with one input and output will be examined. Even such a simple example will give some insight into the two sensitivity transfer functions and, more importantly, an idea of how the three areas under investigation affect the sensitivity of a system.

Consider the plant in Fig. 5 with nominal parameter values of K=10, a=2, and b=1 where "a" is the variable parameter. The states X_1 and X_2 will be defined as outputs of the integrators from right to left or top to bottom, respectively, as applicable for each situation.

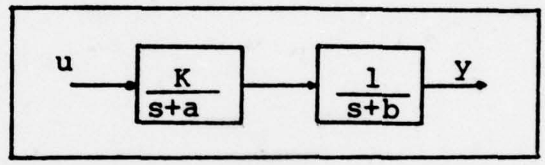


Figure 5. Example Plant

In physical variables the state equations, omitting the time designators for brevity, are

$$\dot{\underline{x}} = \begin{bmatrix} -1 & 1 \\ 0 & -a \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 10 \end{bmatrix} u \tag{26}$$

and

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \underline{x} \tag{27}$$

The implementation of state-variable feedback is shown in Laplace-transformed form in Fig. 6. Using block diagram

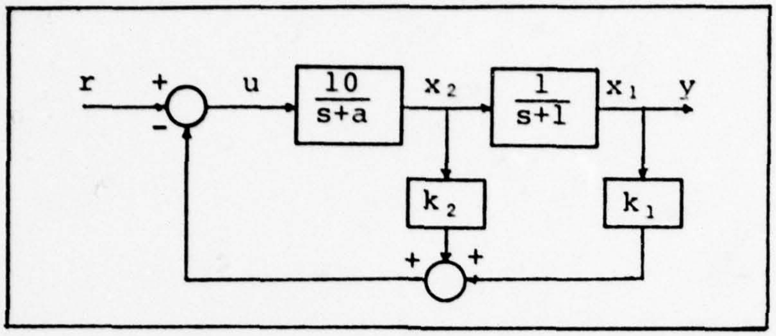


Figure 6. Physical Variable Feedback

reduction techniques the control ratio is found to be

$$\frac{y}{r} = \frac{10}{s^2+(1+a+10k_2)s+(a+10k_2+10k_1)} \tag{28}$$

or, evaluated at the nominal parameter value, a=2,

$$\frac{y}{r} = \frac{10}{s^2+(3+10k_2)s+(2+10k_2+10k_1)} \tag{29}$$

To generate some sort of performance specifications, suppose it is desired that the system have zero steady-state error for a step input. Then $2+10k_2+10k_1=10$ and the control ratio is

$$\frac{y}{r} = \frac{10}{s^2+(3+10k_2)s+10} \tag{30}$$

Now the pole placement is a function only of k_2 with k_1 being determined by $2+10k_2+10k_1=10$. Four cases of pole placement will be investigated. Hereafter, "plant poles" are considered to be the closed-loop poles of the plant.

Case 1. Plant poles: -2, -5; physical variables.

The plant poles will be shifted from -1, -2 to -2, -5 to provide faster transient response decay. The characteristic equation is

$$(s+2)(s+5) = s^2+7s + 10 = 0 \qquad (31)$$

Equating the plant characteristic equation (denominator of Eq (30)) with Eq (31) yields $k_2=0.4$ and thus, $k_1=0.4$.

Case 2. Plant poles: $-2.235\pm j2.235$; physical variables.

The plant poles will be placed at $-2.235\pm j2.235$ to give a slightly oscillatory response with a damping ratio of 0.707. The characteristic equation is

$$(s+2.235+j2.235)(s+2.235-j2.235) = s^2+4.47s + 10 = 0 \qquad (32)$$

with $k_2= 0.1472$ and $k_1= 0.6528$.

Case 3. Plant poles: -1, -10; physical variables.

For a larger shift to the left, poles are placed at -1, -10. Because of its rapid transient decay, the pole at -10 is considered to be nondominant. The characteristic equation becomes

$$(s+1)(s+10) = s^2+11s + 10 = 0 \qquad (33)$$

making $k_2 = 0.8$ and $k_1 = 0.0$.

Case 4. Plant poles: -3.125, -3.2; physical variables.

Finally, the two poles are placed close together

at -3.125 and -3.2. According to Singer (Ref 4:88), this situation has a high eigenvalue sensitivity. This test will help determine the extent, if any, eigenvalue sensitivity affects system sensitivity. The resulting characteristic equation is

$$(s+3.125)(s+3.2) = s^2 + 6.325s + 10 = 0 \qquad (34)$$

forcing $k_2 = 0.3325$ and $k_1 = 0.4675$.

Three observer designs will be considered, each having judiciously-placed poles. It is hoped that this exercise will provide some guidance for observer pole selection for minimizing sensitivity to plant parameter variations.

Observer I. Poles: -6, -6; physical variables.

Double poles are arbitrarily selected at -6. For all but Case 3 above this choice places the observer poles to the left of the plant poles so that the observer reaches steady state before the plant. Analysis of Case 3 results should suggest whether or not this requirement is valid. With a characteristic polynomial of $s^2+12s+36$ and $T=I$, the observer dynamics matrix, from Eq (21), is

$$D = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} = \begin{bmatrix} -1-\nu_1 & 1 \\ -\nu_2 & -2 \end{bmatrix} \qquad (35)$$

The characteristic polynomial of D is found by equating the determinant of [sI-D] to zero, or

$$s^2+(3+\nu_1)s+(2+2\nu_1+\nu_2) = 0 \qquad (36)$$

Equating the coefficients of s in Eq (36) to those of the desired characteristic polynomial, $s^2+12s+36$, V and D are found to be

$$V_1 = \begin{bmatrix} 9 \\ 16 \end{bmatrix} \qquad (37)$$

$$D_1 = \begin{bmatrix} -10 & 1 \\ -16 & -2 \end{bmatrix} \qquad (38)$$

Observer II. Poles: -12, -15; physical variables.

By selecting poles to the left of -10, the observer dynamics will always have reached steady state before the plant. A choice of -12, -15 accomplishes this objective plus it provides a large separation between plant and observer poles for Cases 1, 2, and 4. With a characteristic polynomial of $s^2+27s+180$, V and D are computed, as before with Observer I, to be

$$V_2 = \begin{bmatrix} 24 \\ 130 \end{bmatrix} \qquad (39)$$

$$D_2 = \begin{bmatrix} -25 & 1 \\ -130 & -2 \end{bmatrix} \qquad (40)$$

Observer III. Poles: -3, -4; physical variables.

For the third trial, observer poles will be located in the vicinity of the plant poles to allow some interaction between the two transient responses. Roots

of -3, -4 generate a characteristic polynomial of $s^2+7s+12$, making

$$V_3 = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \tag{41}$$

$$D_3 = \begin{bmatrix} -5 & 1 \\ -2 & -2 \end{bmatrix} \tag{42}$$

In phase-variable form the state equations for the plant in Fig. 5 are derived from the describing differential equation to be

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ -a & -a-1 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{43}$$

$$y = [10 \quad 0]\ \underline{x} \tag{44}$$

The block diagram for this system is given in Fig. 7. The variables to be fed back are not the same x_1 and x_2 as before; therefore, a new relationship between the control ratio and feedback coefficients is necessary. Equation (15) can be solved for $\underline{X}$ which is substituted into Eq (13) to yield the general matrix form of the control ratio

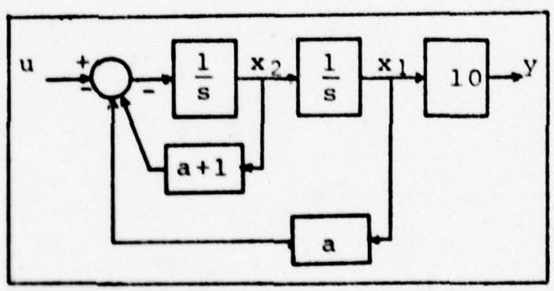


Figure 7. Phase Variable Plant

$$\frac{Y}{r} = C[sI-A+BK]^{-1}B \tag{45}$$

Substituting the corresponding matrices of Eqs (43) and (44) into Eq (45) yields

$$\frac{y}{r} = \frac{10}{s^2+(k_2+3)s+(k_1+2)} \tag{46}$$

which is the desired control ratio as a function of phase-variable feedback coefficients.

The necessary feedback coefficients are to be calculated for pole placements identical to those presented in the physical variable development using the same procedure discussed there. It is obvious that for zero steady-state step error $k_1=8$ and is independent of pole placement.

Case 5. Plant poles: -2, -5; phase variables.

The characteristic equation was $s^2+7s+10=0$, so that $k_2=4$.

Case 6. Plant poles: $-2.235\pm j\ 2.235$; phase variables.

The characteristic equation was $s^2+4.471s+10=0$, making $k_2=1.471$.

Case 7. Plant poles: -1, -10; phase variables.

The characteristic equation was $s^2+11s+10=0$, or $k_2=8$.

Case 8. Plant poles: -3.125, -3.2; phase variables.

The characteristic equation was $s^2+6.325s+10=0$, thus $k_2=3.325$.

Observer IV. Poles: -6, -6; phase variables.

Solving Eq (21) at nominal parameter values and equating the coefficients of the characteristic polynomial of D with the coefficients of the desired characteristic equation, $s^2+12s+36=0$, results in

$$V_4 = \begin{bmatrix} 0.9 \\ 0.7 \end{bmatrix} \tag{47}$$

$$D_4 = \begin{bmatrix} -9 & 1 \\ -9 & -3 \end{bmatrix} \tag{48}$$

Observer V. Poles: -12, -15; phase variables.

Repeating the procedures for determining V and D with the desired observer characteristic equation of $s^2+27s+180=0$ gives

$$V_5 = \begin{bmatrix} 2.4 \\ 10.6 \end{bmatrix} \tag{49}$$

$$D_5 = \begin{bmatrix} -24 & 1 \\ -108 & -3 \end{bmatrix} \tag{50}$$

Observer VI. Poles: -3, -4; phase variables.

For the desired characteristic equation of $s^2+7s+12=0$, V and D were computed, as before, to be

$$V_6 = \begin{bmatrix} 0.4 \\ -0.2 \end{bmatrix} \tag{51}$$

$$D_6 = \begin{bmatrix} -4 & 1 \\ 0 & -3 \end{bmatrix} \tag{52}$$

To describe the basic plant in Jordan canonical form, the plant transfer function must be rewritten in its partial fraction expansion form as

$$y = \frac{10}{1-a}\left(\frac{1}{s+a} - \frac{1}{s+1}\right) u \tag{53}$$

The simulation diagram for Eq (53) is given in Fig. 8. The new state variables x_1 and x_2 are not the same as in either of the preceding state-variable forms. The state equations can be derived from Fig. 8 to be

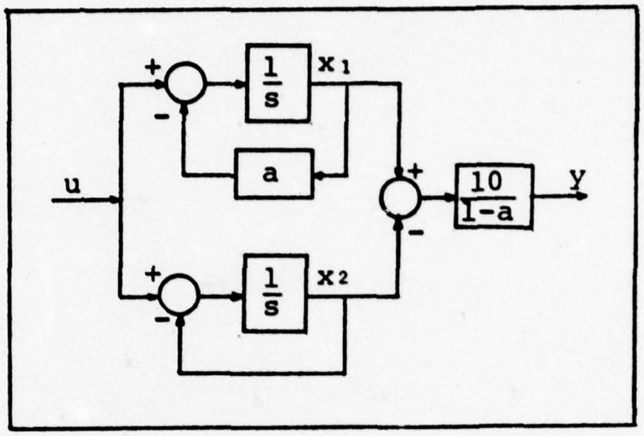


Figure 8. Jordan Form Plant

$$\dot{\underline{x}} = \begin{bmatrix} -a & 0 \\ 0 & -1 \end{bmatrix} \underline{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \tag{54}$$

$$y = \begin{bmatrix} \frac{10}{1-a} & \frac{-10}{1-a} \end{bmatrix} \underline{x} \tag{55}$$

To obtain the control ratio as a function of the feedback coefficients in this case, the A, B, and C matrices of Eqs (54) and (55) are substituted into Eq (45). The result is

$$\frac{y}{r} = \frac{10}{s^2+(3+k_1+k_2)s+(2+k_1+2k_2)} \tag{56}$$

For zero steady-state step error, $2+k_1+2k_2=10$. The feedback coefficients are computed for each case by equating like coefficients of s in Eq (56) with those of the desired characteristic equations.

Case 9. Plant poles: -2, -5; Jordan canonical variables.

The characteristic equation was $s^2+7s+10=0$; $k_1=0$, $k_2=4$.

Case 10. Plant poles: $-2.235\pm j2.235$; Jordan canonical variables.

The characteristic equation was $s^2+4.471s+10=0$; $k_1=-5.056$, $k_2=6.528$.

Case 11. Plant poles: -1, -10; Jordan canonical variables.

The characteristic equation was $s^2+11s+10=0$; $k_1=8$, $k_2=0$.

Case 12. Plant poles: -3.125, -3.2; Jordan canonical variables.

The characteristic equation was $s^2+6.325s+10=0$; $k_1=-1.35$, $k_2=4.675$.

Using the same procedure described in the physical and phase variable cases, the following V and D matrices were computed for the Jordan form observers:

Observer VII. Poles: -6, -6; Jordan canonical variables.

$$V_7 = \begin{bmatrix} 1.6 \\ 2.5 \end{bmatrix} \tag{57}$$

$$D_7 = \begin{bmatrix} 14 & -16 \\ 25 & -26 \end{bmatrix} \tag{58}$$

Observer VIII. Poles: -12, -15; Jordan canonical variables.

$$V_8 = \begin{bmatrix} 13 \\ 15.4 \end{bmatrix} \tag{59}$$

$$D_8 = \begin{bmatrix} 128 & -130 \\ 154 & -155 \end{bmatrix} \tag{60}$$

Observer IX. Poles: -3, -4; Jordan canonical variables.

$$V_9 = \begin{bmatrix} 0.2 \\ 0.6 \end{bmatrix} \tag{61}$$

$$D_9 = \begin{bmatrix} 0 & -2 \\ 6 & -7 \end{bmatrix} \tag{62}$$

Sensitivity transfer functions were computed for each of the 48 combinations of state-variable representations, pole placements, and observer designs as displayed in Fig. 9. The algorithm used to perform the necessary matrix inversions for Eqs (20) and (25) is presented in Appendix A. As a means for comparing relative sensitivities of the state-variable feedback model with the observer feedback models, frequency responses of the sensitivity transfer functions were plotted from 0-5 radians per second. This range is considered to be

representative of the operational bandwidth of practical control systems. Figs. 10-17 display the absolute value of the magnitudes of the sensitivity functions for all the cases. It was noticed that the results for physical variables and phase variables were always identical (to about four decimal places); therefore, only one plot was used to represent the two data runs for each similar test.

Plant Poles	Physical Variables	Phase Variables	Jordan Canonical Variables
-2,-5	<u>Case 1</u> Direct feedback Observer I (-6,-6) Observer II (-12,-15) Observer III (-3, -4)	<u>Case 5</u> Direct feedback Observer IV (-6,-6) Observer V (-12,-15) Observer VI (-3,-4)	<u>Case 9</u> Direct feedback Observer VII (-6,-6) Observer VIII (-12,-15) Observer IX (-3,-4)
$-2.235 \pm j2.235$	<u>Case 2</u> Direct feedback Observer I (-6, -6) Observer II (-12,-15) Observer III (-3,-4)	<u>Case 6</u> Direct feedback Observer IV (-6,-6) Observer V (-12,-15) Observer VI (-3,-4)	<u>Case 10</u> Direct feedback Observer VII (-6,-6) Observer VIII (-12,-15) Observer IX (-3,-4)
-1,-10	<u>Case 3</u> Direct feedback Observer I (-6,-6) Observer II (-12,-15) Observer III (-3,-4)	<u>Case 7</u> Direct feedback Observer IV (-6,-6) Observer V (-12,-15) Observer VI (-3,-4)	<u>Case 11</u> Direct feedback Observer VII (-6,-6) Observer VIII (-12,-15) Observer IX (-3,-4)
-3.125, -3.2	<u>Case 4</u> Direct feedback Observer I (-6,-6) Observer II (-12,-15) Observer III (-3,-4)	<u>Case 8</u> Direct feedback Observer IV (-6,-6) Observer V (-12,-15) Observer VI (-3,-4)	<u>Case 12</u> Direct feedback Observer VII (-6,-6) Observer VIII (-12,-15) Observer IX (-3,-4)

Figure 9. Matrix of Test Cases

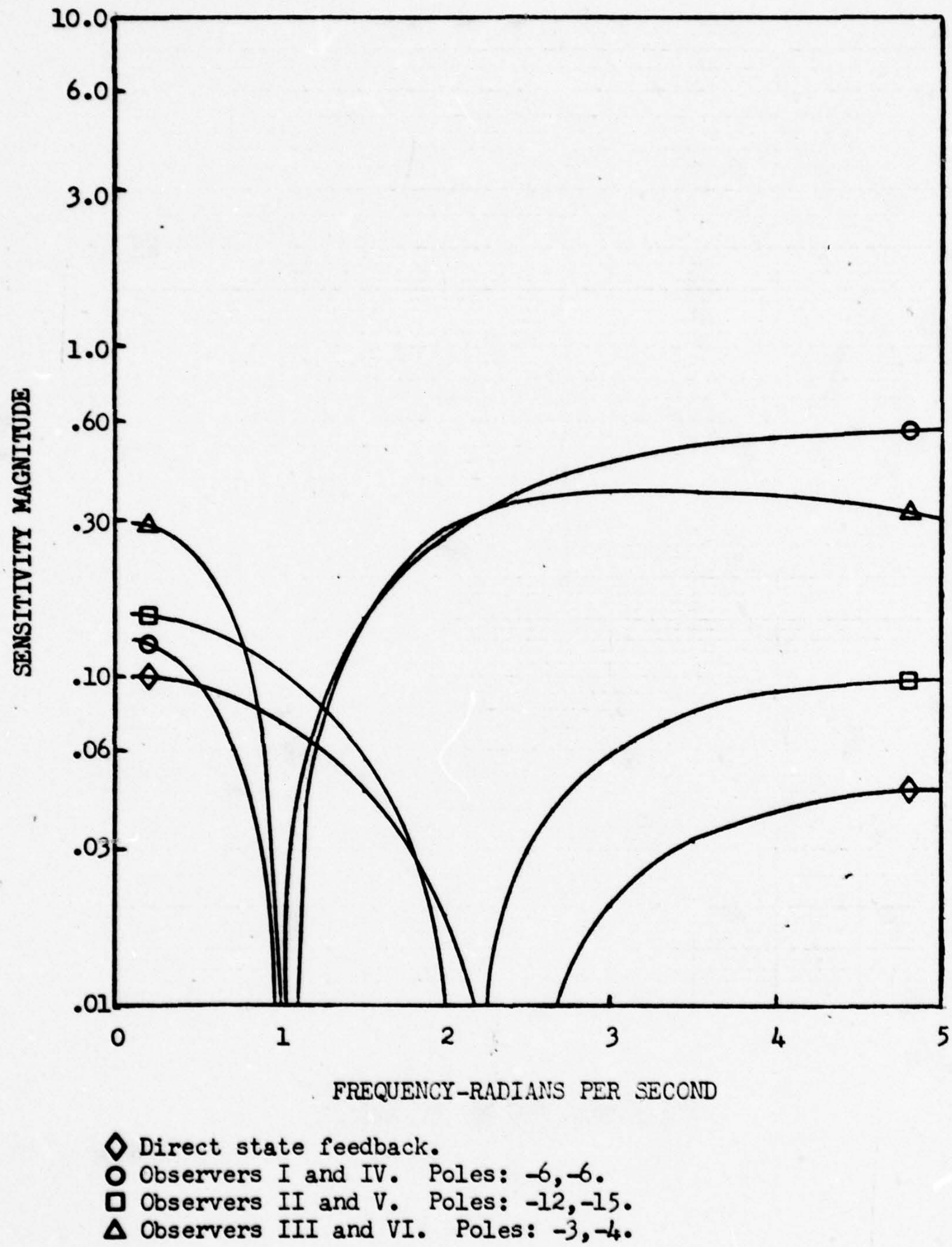
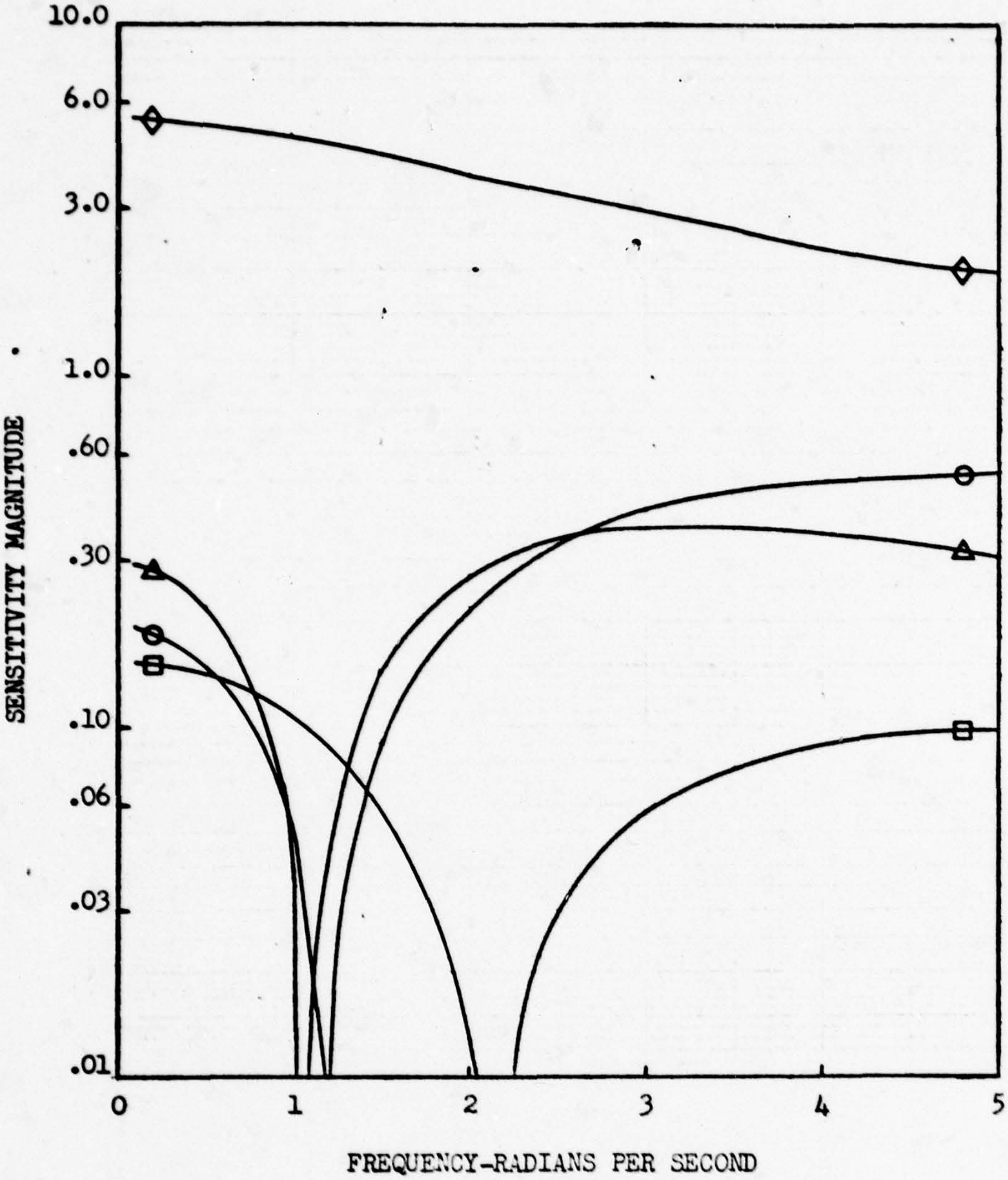
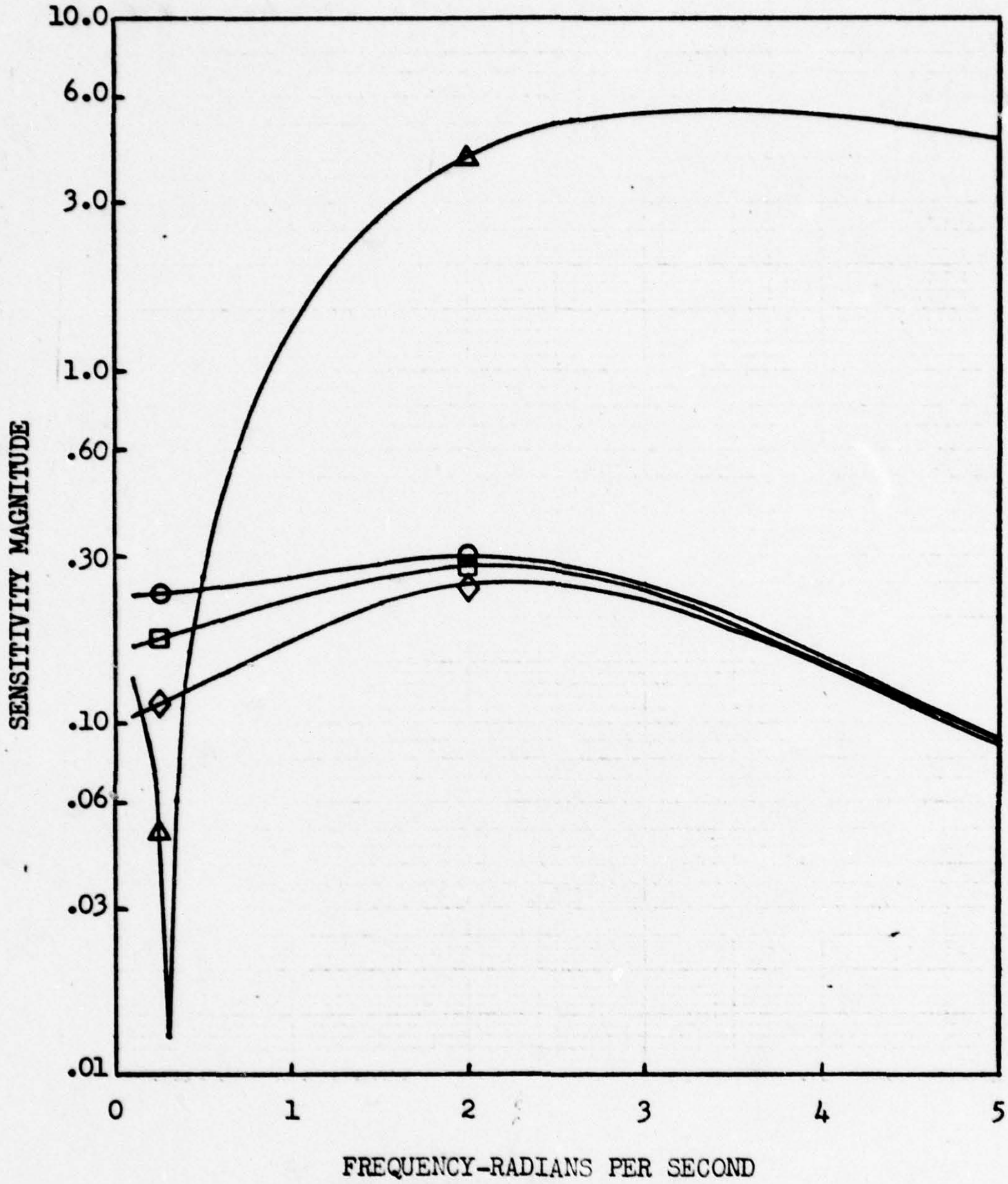


Figure 10. Case 1. Plant Poles: -2,-5; Physical Variables.
Case 5. Plant Poles: -2,-5; Phase Variables.



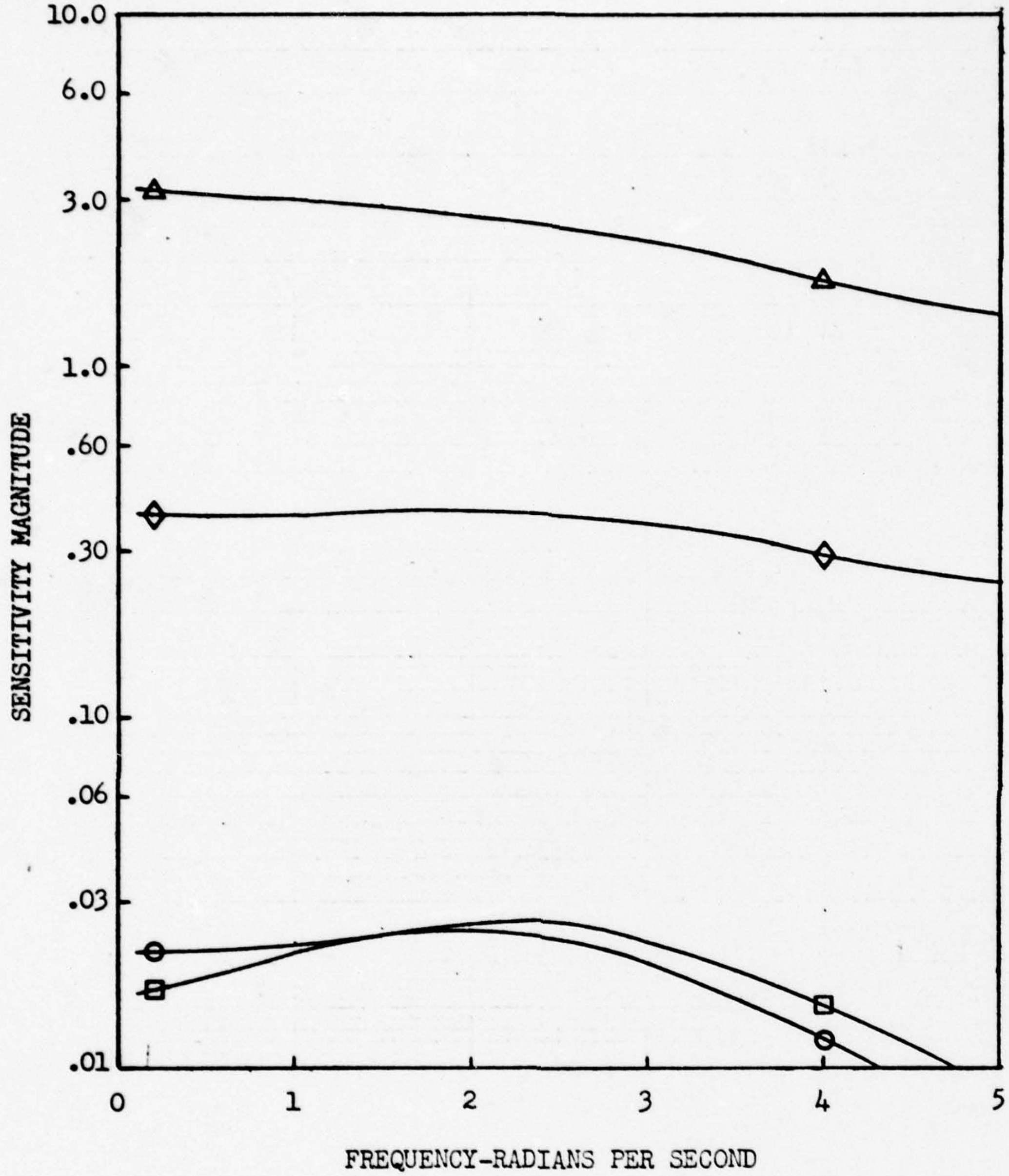
◇ Direct state feedback.
○ Observer VII. Poles: -6,-6.
□ Observer VIII. Poles: -12,-15.
△ Observer IX. Poles: -3,-4.

Figure 11. Case 9. Plant Poles: -2,-5; Jordan Canonical Variables.



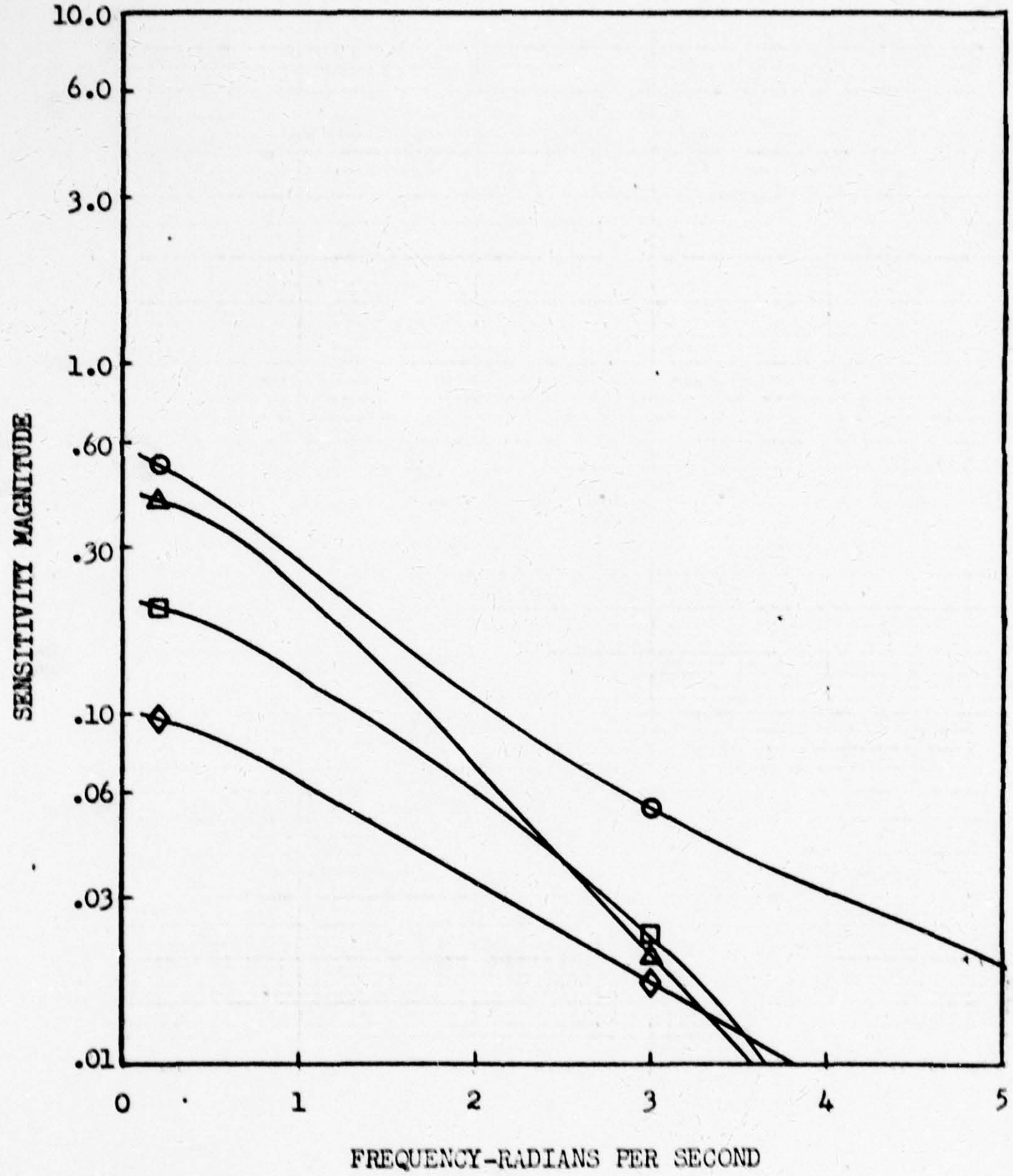
◇ Direct state feedback.
○ Observers I and IV. Poles: -6,-6.
□ Observers II and V. Poles: -12,-15.
△ Observers III and VI. Poles: -3,-4.

Figure 12. Case 2. Plant Poles: $-2.235 \pm j2.235$; Physical Variables.
Case 6. Plant Poles: $-2.235 \pm j2.235$; Phase Variables.



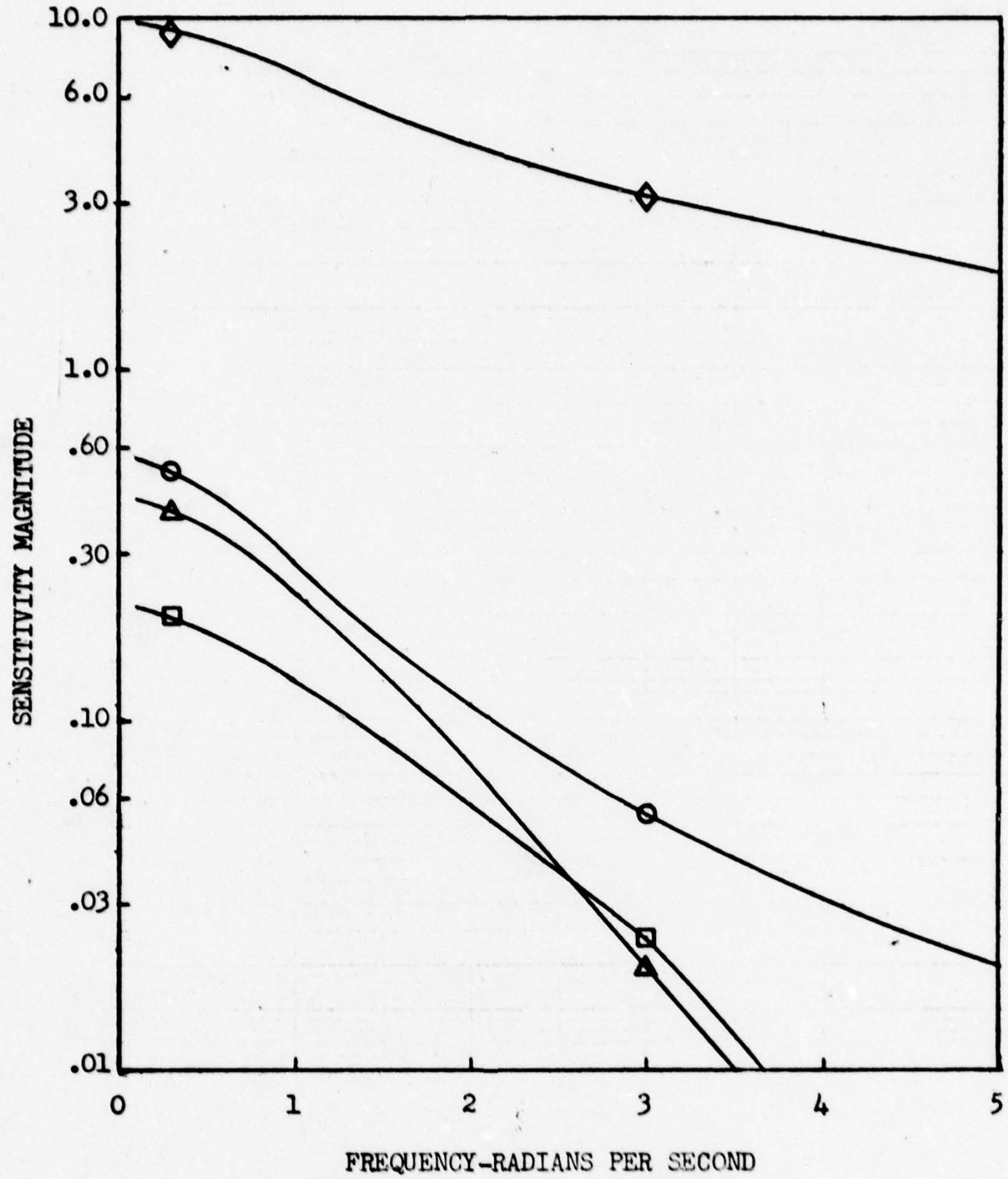
◇ Direct state feedback.
○ Observer VII. Poles: -6,-6.
□ Observer VIII. Poles: -12,-15.
△ Observer IX. Poles: -3,-4.

Figure 13. Case 10. Plant Poles: $-2.235 \pm j2.235$; Jordan Canonical Variables.



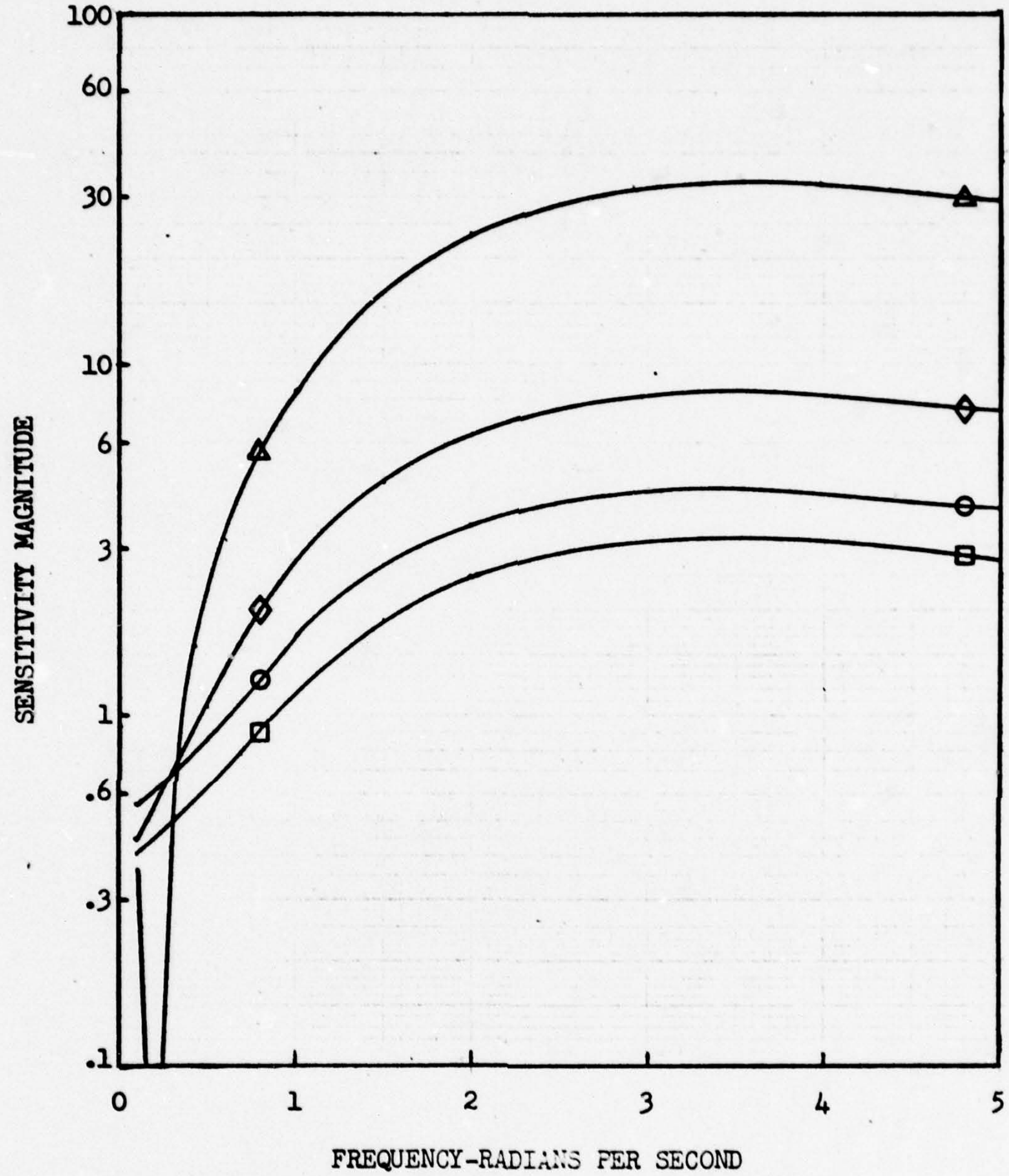
◇ Direct state feedback.
○ Observers I and IV. Poles: -6,-6.
□ Observers II and V. Poles: -12,-15.
△ Observers III and VI. Poles: -3,-4.

Figure 14. Case 3. Plant Poles: -1,-10; Physical Variables.
Case 7. Plant Poles: -1,-10; Phase Variables.



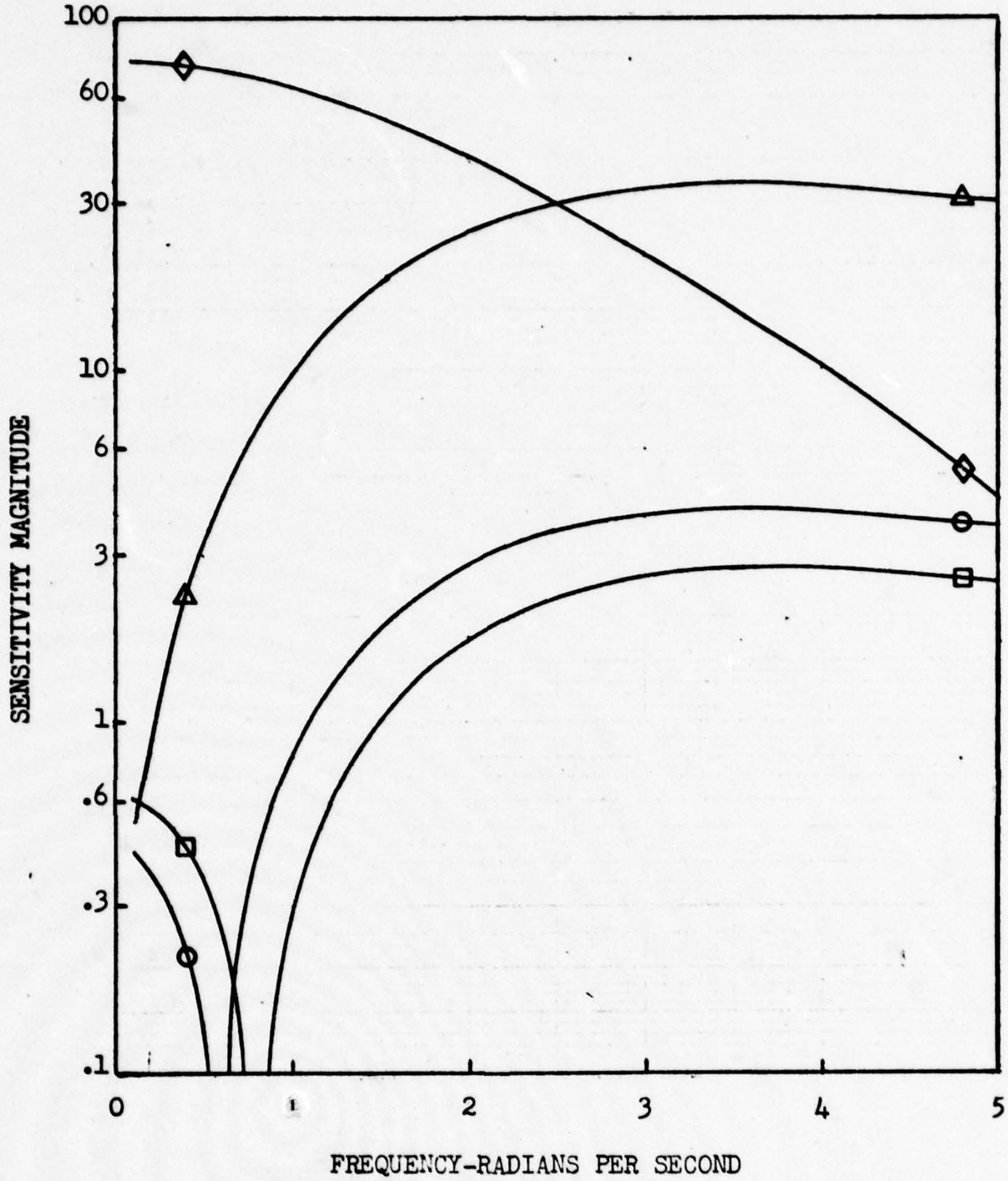
◇ Direct state feedback.
○ Observer VII. Poles: -6,-6.
□ Observer VIII. Poles: -12,-15.
△ Observer IX. Poles: -3,-4.

Figure 15. Case 11. Plant Poles: -1,-10; Jordan Canonical Variables.



◇ Direct state feedback.
○ Observers I and IV. Poles: -6,-6.
□ Observers II and V. Poles: -12,-15.
△ Observers III and VI. Poles: -3,-4.

Figure 16. Case 4. Plant Poles: -3.125,-3.2; Physical Variables.
Case 8. Plant Poles: -3.125,-3.2; Phase Variables.



◇ Direct state feedback.
○ Observer VII. Poles: -6,-6.
□ Observer VIII. Poles: -12,-15.
△ Observer IX. Poles: -3,-4.

Figure 17. Case 12. Plant Poles: -3.125,-3.2; Jordan Canonical Variables.

IV. Results and Discussion

To avoid confusion over notation of the different test cases in this section, an explicit coding is needed to distinguish the three aspects of the cases: state-variable representation, plant poles, and observer pole locations. Since the results for physical variables and phase variables were identical in all cases studied, it is necessary to refer only to one of the two; say, phase variables. Then the following notation is adopted: (a; b, c; d, e) where a is the type of state variable (P for phase variables; J for Jordan canonical variables); b and c are the plant pole locations (for complex conjugate poles b and c are the real and imaginary parts, respectively); d and e are the observer pole locations. If d and e are not listed, direct state feedback is implied. When observers are individually mentioned by their respective numbers, the type of state variable representation and observer poles will immediately follow as in IV (P; -6, -6).

Data Summary

Case 5. (P; -2, -5), (P; -2, -5; -6, -6), (P; -2, -5; -12, -15), (P; -2, -5; -3, -4) and Case 9. (J; -2, -5; -6, -6), (J; -2, -5; -12, -15), (J; -2, -5; -3, -4), (J; -2, -5). For the phase-variable representation (Fig. 10), all three observers exhibited greater sensitivity magnitudes than did the plant with direct state feedback; in Jordan form

(Fig. 11) the opposite was true with an increase of two orders of magnitude in the direct state feedback. The observer sensitivities were independent of state-variable representation. Although observers V (P; -12, -15) and VIII (J; -12, -15) were the least sensitive, followed by observers VI (P; -3, -4) and IX (J; -3, -4), then IV (P; -6, -6) and VII (J; -6, -6) in increasing order, none had a sensitivity magnitude greater than 0.54 in the bandwidth of interest; thus the system changes by only half the magnitude of a parameter variation. Observers IV (P; -6, -6), VI (P; -3, -4), VII (J; -6, -6) and IX (J; -3, -4) displayed extreme insensitivity to parameter variations at about one radian per second; observers V (P; -12, -15) and VIII (J; -12, -15) were insensitive near two radians per second, as was the direct state feedback function in phase-variable form. In Jordan form the direct state feedback sensitivity was not affected by this phenomenon. The algorithm used to generate data points considered only the absolute value of the sensitivity function. In expanding the $[sI-A]^{-1}$ terms, some of the components were sufficiently negative to impart a negative sign to the sensitivity function for some frequencies. The physical implication is that the system output deviates in the opposite direction of the parameter variation. It also means there is a frequency for which the system has zero sensitivity to parameter variations.

Case 6. (P; -2.235, ±2.235), (P; -2.235, ±2.235; -6, -6), (P; -2.235, ±2.235; -12, -15), (P; -2.235, ±2.235; -3, -4) and Case 10. (J; -2.235, ±2.235), (J; -2.235, ±2.235; -6, -6), (J; -2.235 ±2.235; -12, -15), (J; -2.235, ±2.235; -3, -4). In Fig. 12 observers IV (P; -6, -6) and V (P; -12, -15) very closely approximated the sensitivity function for direct feedback. Observer VI (P; -3, -4) exhibited the sign inversion, previously discussed, at about 0.3 radians per second, becoming extremely sensitive with increasing frequency. In Fig. 13, observer IX (J; -3, -4) was even more sensitive to variations, being over two orders of magnitude above observers VII (J; -6, -6) and VIII (J; -12, -15). Again, in Jordan form, the direct state feedback system sensitivity was an order of magnitude greater than it was in phase-variable form. Observers IV (P; -6, -6), V (P; -12, -15), VII (J; -6, -6), and VIII (J; -12, -15) were all very similar with peak sensitivities of 0.3.

Case 7. (P; -1, -10), (P; -1, -10; -6, -6), (P; -1, -10; -12, -15), (P; -1, -10; -3, -4) and Case 11. (J; -1, -10), (J; -1, -10; -6, -6), (J; -1, -10; -12, -15), (J; -1, -10; -3, -4). The only difference between Figs. 14 and 15 was the sensitivity of the direct state feedback system. In phase-variable form, direct state feedback was the least sensitive implementation whereas in Jordan form, it was increased by two orders of magnitude; observer

performances were identical for the two state-variable representations.

<u>Case 8. (P; -3.125, -3.2), (P; -3.125, -3.2; -6, -6), (P; -3.125, -3.2; -12, -15), (P; -3.125, -3.2; -3, -4) and Case 12. (J; -3.125, -3.2), (J; -3.125, -3.2; -6, -6), (J; -3.125, -3.2; -12, -15), (J; -3.125, -3.2; -3, -4)</u>. In these cases all systems exhibited very high sensitivities to parameter variations with the observers being basically independent of state-variable representation. Observer VI (P; -3, -4) underwent a sign inversion at 0.2 radian per second; observers VII (J; -6, -6) and VIII (J; -12, -15) experienced sign inversions at 0.55 and 0.8 radian per second, respectively. The direct state feedback system sensitivity was much greater in Jordan form than in phase-variable form.

Analysis

The first question posed in developing this study concerned the effects of different state-variable representations upon system sensitivity. The data show that there is no difference in sensitivity between physical and phase variables and that the Jordan form is always more sensitive than the former two representations. This does not correlate with eigenvalue sensitivity concepts. Reference 4 demonstrates that unity eigenvalue sensitivity is always obtained with the Jordan form if the eigenvalues are real whereas, with phase variables, the eigenvalue sensitivity can

range from less than one to infinity, depending upon the pole locations. The following phase-variable eigenvalue sensitivities (S) were computed for the examples investigated: for plant poles of -2, -5: S=1, 2, respectively; for plant poles of $-2.235 \pm j\ 2.235$: S=0.88, 0.88, respectively; for plant poles of -1, -10: S=0.22, 1.22, respectively; and finally, for plant poles of -3.125, -3.2: S=55, 56, respectively (Ref 4:88). The phase-variable eigenvalue sensitivities for the first three sets of plant poles are all in the vicinity of S=1, which is the Jordan form eigenvalue sensitivity, and yet the system sensitivities for those cases vary greatly between the two different state-variable representations. It is not understood why this apparent discrepancy occurs. It would seem that low system sensitivity would imply low eigenvalue sensitivity and conversely, but the converse does not hold here. The observer sensitivities, on the other hand, did not vary much from one state-variable representation to another; only the direct state feedback sensitivity functions were affected by the Jordan form.

The second question was that of pole placement effects upon system sensitivity. The extent of desired pole placement definitely can change the system sensitivity. Attempting to place poles close together drastically increases parameter variation sensitivity; creation of complex conjugate poles does not have adverse sensitivity effects. Low system sensitivity is also achieved by pairing

a pole near the origin with one deeper into the left half-plane. The magnitudes of the feedback gains required to achieve the pole placements do not appear to affect the system sensitivities as seen by the duplicate results for physical and phase variables for which the feedback coefficients differed by an order of magnitude. There were no obvious adverse effects from the negative k_1 coefficients (which create positive, unstable feedback) in several of the Jordan form cases as all exhibited the same high sensitivity characteristics.

The final area of investigation, that of effects of observer design upon system sensitivity, revealed a definite relationship between the two factors. The design of observer dynamics is the most important aspect in reducing the sensitivity of an observed plant to internal parameter variations. As mentioned in Chapter II, literature on observers states only that the observer poles should be placed to the left of all plant poles for good performance. Analysis of the test cases in this report indicates that such a guideline may be too restrictive upon observer design. In several of the cases involving plant poles of -1, -10, placing observer poles to the left of the dominant pole at -1, but to the right of the nondominant pole at -10, achieved not only low system sensitivity but identical performance in all three state-variable forms. In general, though, placing the observer poles to the left of all plant poles does result in a lower system sensitivity.

V. Conclusions and Recommendations

Conclusions

The selection of state variables to be utilized in system analysis does affect the outcome. Physical variables are the best choice when applicable because they represent measurable quantities of interest within a system. If physical variables are not realizable for some reason, such as when a derivative of the input (zero) appears with the n-th state, phase variables yield identical sensitivity results. Jordan canonical variables should be avoided because of the relatively high system sensitivity that results from this form.

In placing closed-loop poles through state-variable feedback, system sensitivity is reduced when the poles are separated from each other, especially if one pole is non-dominant. This result is also seen in Chapter 12 of Reference 1. Observer poles should be placed to the left of all dominant plant poles as a minimal requirement, and to the left of all plant poles, if practical, for low system sensitivity to parameter variations.

The "bottom line" is that the observer can be designed so that the resulting system sensitivity is as low as that of direct state feedback, or even lower in many cases.

Recommendations

Several areas for further research were identified

during this study. The observers considered were full-order types; that is, the entire state vector was reconstructed for feedback purposes. It is possible to use available states for feedback and design a reduced-order observer to generate only the inaccessible states. The observer has fewer states and may reduce system sensitivity since there is less probability of interaction between plant parameter variations and the observer. There exists a trade-off between acceptable system sensitivity and the cost/availability of sensors to pick off plant states. An investigation should be conducted to determine the sensitivity aspects of reduced-order observers.

Equation (10) expresses the change in output magnitude due to a parameter variation, but does not address corresponding phase angle changes. In the data summary of Chapter IV, sign changes were credited with causing zero parameter sensitivities at certain frequencies. A question arises as to whether or not it is possible to predict conditions under which the "phase angle sensitivity" ($\partial\phi/\partial p$) causes a sign change in the sensitivity magnitude. If so, it may be possible to alter some design parameters to create zero sensitivity at frequencies of known external disturbances. Such a capability would be a most useful addition to control theory.

Another area is that of plants with multiple inputs and outputs. Luenberger states that the problem of designing an observer for a plant with multiple outputs involves

a series of observers, one for each output (Ref 6:190). Since the output of each observer is intimately tied to the plant, and hence to each other observer through cross-coupling, there may be some interesting sensitivity consequences to a plant parameter variation.

Bibliography

1. D'Azzo, John J. and Constantine H. Houpis. Linear Control System Analysis and Design: Conventional and Modern. New York: McGraw-Hill Book Co., 1975.

2. Melsa, James L. and Donald G. Schultz. Linear Control Systems. New York: McGraw-Hill Book Co., 1969.

3. Luenberger, David G. "Observing the State of a Linear System." IEEE Transactions on Military Electronics, MIL-8: 74-80 (April 1964).

4. Singer, R. A. "Selecting State Variables to Minimize Eigenvalue Sensitivity of Multivariable Systems." Automatica, 5:85-93 (1969).

5. Mantey, Patrick E. "Eigenvalue Sensitivity and State-Variable Selection." IEEE Transactions on Automatic Control, AC-13: 263-269 (June 1968).

6. Luenberger, D. G. "Observers for Multivariable Systems." IEEE Transactions on Automatic Control, AC-11: 190-197 (April 1966).

7. -----. "An Introduction to Observers." IEEE Transactions on Automatic Control, AC-16: 596-602 (December 1971).

8. Newmann, M. M. "Design Algorithm for Minimal-Order Luenberger Observers." Electronics Letters, 5:390-392 (August 1969).

9. Maeda, Hajime and Hirokazu Hino. "Design of Optimal Observers for Linear Time-Invariant Systems." International Journal of Control, 19:993-1000 (May 1974).

10. Anderson, B. D. O. and J. B. Moore. Linear Optimal Control. Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1971.

11. Reid, J. Gary, et al. "An Algebraic Representation of Parameter Sensitivity in Linear Time-invariant Systems." Journal of the Franklin Institute, 301:123-141 (January-February 1976).

12. DeRusso, Paul M., et al. State Variables for Engineers. New York: John Wiley & Sons, Inc., 1965.

13. Reid, J. Gary. "Sensitivity Operators and Associated System Concepts for Linear Dynamic Systems." Air Force Avionics Lab. Tech. Rep. AFAL-TR 76-118, Wright-Patterson Air Force Base, Ohio, July 1976.

APPENDIX A

Algorithm for Computing Sensitivity Transfer Functions

The only difficult task in computing the sensitivity transfer functions, Eqs (20) and (25), is inverting the 2n x 2n matrix [sI-A'] and the 4n x 4n matrix [sI-A"]. Obviously, for n>1 this becomes a formidable task for hand calculation and computer routines inherently solve numerical matrix inversions. Reference 11 presents a convenient scheme for accomplishing this task, based upon the fact that

$$[sI-\bar{M}]^{-1} = \mathcal{L}(e^{\bar{M}t}) \tag{63}$$

There are a number of methods available for expanding a matrix exponential, but this scheme is readily adaptable to a digital computer. The four basic steps, each of which is discussed later, are as follows:

1. Construct the Vandermonde matrix, Λ.
2. Compute $\Lambda^{-1} = V$.
3. Construct the component matrices, $\bar{Z}_{k,i}$, of $\bar{M}$.
4. Expand Eq (63) as

$$\mathcal{L}(e^{\bar{M}t}) = \sum_{k=1}^{q} \sum_{i=0}^{m_k-1} \bar{Z}_{k,i} \mathcal{L}(t^i e^{\lambda_k t}) \tag{64}$$

where q is the number of distinct eigenvalues of $\bar{M}$ and m_k is the multiplicity of λ_k in the minimal polynomial of $\bar{M}$. (See Appendix B for a brief discussion of minimal polynomials.)

A useful property of the matrices A' and A" in Eqs (20) and (25), respectively, is that they have the form

$$\bar{M} = \left[\begin{array}{c|c} M & 0 \\ \hline \frac{\partial M}{\partial p} & M \end{array}\right]_{2n \times 2n} \tag{65}$$

where n is the dimension of each partition of $\bar{M}$. The eigenvalues of $\bar{M}$ are simply the eigenvalues of M, each repeated once. The characteristic polynomial of $\bar{M}$ is defined as

$$\Delta(\lambda) = \det(\lambda I - \bar{M}) = \prod_{k=1}^{q} (\lambda - \lambda_k)^{2n_k} \tag{66}$$

The Vandermonde matrix of $\bar{M}$ is constructed from the eigenvalues of $\bar{M}$ as r rows of the form

$$\frac{d^j}{d\lambda^j} [1 \;\lambda_k \;\lambda_k^2 \;\ldots \;\lambda_k^{2n-1}] \tag{67}$$

where $k=1,2,\ldots,r$ and $j=0,1\ldots,2n_k-1$ for r real, distinct eigenvalues and q-r rows of the form

$$\text{Real}\left(\frac{d^j}{d\lambda^j} [1 \;\lambda_k \;\lambda_k^2 \ldots \lambda_k^{2n-1}]\right)$$

$$\text{Imag } \left(\frac{d^j}{d\lambda^j}\,[1\ \lambda_k\ \lambda_k^2 \ldots \lambda_k^{2n-1}]\right) \tag{68}$$

where $k=r+1, r+3, \ldots q-1$ and $j=0,1,\ldots,2n_k-1$ for complex eigenvalues. The odd subscript accounts for the fact that complex eigenvalues occur in conjugate pairs as

$$\lambda_k = \sigma_k + jw_k, \quad \lambda_{k+1} = \sigma_k - jw_k$$

Once the Vandermonde matrix is complete, Step 2 can be accomplished by any matrix inversion routine.

Computation of the components, $\bar{Z}_{k,i}$, of $\bar{M}$ depends intimately upon the form of the Vandermonde matrix for each system. $\bar{Z}_{k,i}$ is, itself, a partitioned matrix of nxn component matrices, $Z_{k,i}$ and $ZP_{k,i}$ of M and $\partial M/\partial p$, respectively; that is,

$$\bar{Z}_{k,i} = \left[\begin{array}{c|c} Z_{k,i} & 0 \\ \hline ZP_{k,i} & Z_{k,i} \end{array}\right] \tag{69}$$

where $Z_{k,i} = \sum_{j=1}^{2n} M^{j-1}\, V_{j,f} \quad k=0,1,\ldots q;\ i=0,1,\ldots,m_k-1$

(70)

and $ZP_{k,i} = \sum_{j=2}^{2n} \left(\frac{\partial M}{\partial p}\right)^{j-1} V_{j,fp}$

$k=0,1,\ldots,q;\ i=0,1,\ldots n_k-1$

(71)

The subscript fp in Eq (71) acts as a counting index. For example, in $ZP_{1,0}$ fp=1; $ZP_{1,1}$ fp=2 and so forth up to

$fp=4n$. It should be noted that fp remains fixed over the entire summation for each component matrix $ZP_{k,i}$. The subscript f in Eq (70) is not so simply stated. It is the index number of the row in the Vandermonde matrix that corresponds to the i-th derivative of the k-th eigenvalue. This is better understood through an example. Assume $n=4$ and that there are four distinct, real eigenvalues of M. Then $q=4$, $r=4$, and $n_1= n_2= n_3= n_4= 1$. The Vandermonde matrix is

$$\Lambda = \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \lambda_1^3 & \lambda_1^4 & \lambda_1^5 & \lambda_1^6 & \lambda_1^7 \\ 0 & 1 & 2\lambda_1 & 3\lambda_1^2 & 4\lambda_1^3 & 5\lambda_1^4 & 6\lambda_1^5 & 7\lambda_1^6 \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & & & & \\ 0 & 1 & 2\lambda_2 & \cdots & & & & \\ 1 & \lambda_3 & \lambda_3^2 & \cdots & & & & \\ 0 & 1 & 2\lambda_3 & \cdots & & & & \\ 1 & \lambda_4 & \lambda_4^2 & \cdots & & & & \\ 0 & 1 & 2\lambda_4 & \cdots & & & & \end{bmatrix} \tag{72}$$

In this case, for $Z_{1,0} \rightarrow f=1$ (the 0-th derivative of λ_1 is row 1 of Λ); $Z_{2,0} \rightarrow f=3$; $Z_{3,0} \rightarrow f=5$; $Z_{4,0} \rightarrow f=7$. If, say, λ_1, is repeated once, then $q=3=r$ and $n_1=2$, $n_2=n_3=1$ and the Vandermonde matrix is

$$\Lambda = \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \lambda_1^3 & \lambda_1^4 & \lambda_1^5 & \lambda_1^6 & \lambda_1^7 \\ 0 & 1 & 2\lambda_1 & 3\lambda_1^2 & 4\lambda_1^3 & 5\lambda_1^4 & 6\lambda_1^5 & 7\lambda_1^6 \\ 0 & 0 & 2 & 6\lambda_1 & 12\lambda_1^2 & 20\lambda_1^3 & 30\lambda_1^4 & 42\lambda_1^5 \\ 0 & 0 & 0 & 6 & 24\lambda_1 & 60\lambda_1^2 & 120\lambda_1^3 & 210\lambda_1^4 \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & & & & \\ 0 & 1 & 2\lambda_2 & \cdots & & & & \\ 1 & \lambda_3 & \lambda_3^2 & \cdots & & & & \\ 0 & 1 & 2\lambda_3 & \cdots & & & & \end{bmatrix} \tag{73}$$

For $Z_{1,0} \rightarrow f=1$; $Z_{1,1} \rightarrow f=2$ (the first derivative of λ_1 is row 2 of Λ); $Z_{2,0} \rightarrow f=5$; $Z_{3,0} \rightarrow f=7$. The pattern holds for any case. Once $Z_{k,i}$ and $ZP_{k,i}$ have been computed, the component matrices $\bar{Z}_{k,i}$ are formed as shown in Eq (69). It should be noted that for each $\bar{Z}_{k,i}$ there is a $ZP_{k,i}$ but not necessarily a $Z_{k,i}$. In such cases the missing $Z_{k,i}$ is an nxn zero matrix.

The expansion of Eq (60) is straightforward once the component matrices have been defined. The exponential expansion is substituted into the sensitivity transfer function equation. The remaining operations are simple matrix algebra.

APPENDIX B

Minimal Polynomial

The characteristic polynomial of a matrix M is defined as

$$\Delta(\lambda) = \det(\lambda I - M) = \prod_{k=1}^{q} (\lambda - \lambda_k)^{n_k} \tag{74}$$

where q is the number of distinct eigenvalues of M and n_k is the multiplicity of λ_k in M. The adjoint matrix of M is defined to be the transpose of the matrix of cofactors of M (see, for example, Ref 12:211). The largest factor that is common to all elements of the adjoint matrix of $(\lambda I - M)$ is called $g(\lambda)$ and is at least unity; that is,

$$\operatorname{adj}(\lambda I - M) = \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ a_{n,1} & \cdots & a_{n,n} \end{bmatrix} = g(\lambda) \begin{bmatrix} b_{1,1} & \cdots & b_{1,n} \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ b_{n,1} & \cdots & b_{n,n} \end{bmatrix} \tag{75}$$

The minimal polynomial of M is calculated from

$$m(\lambda) = \Delta(\lambda)/g(\lambda) = \prod_{k=1}^{q} (\lambda - \lambda_k)^{m_k} \tag{76}$$

where m_k, $n_k \geq m_k \geq 1$, is the multiplicity of λ_k in the minimal polynomial of M (Ref 13:65).

Vita

Dennis Larry Hamme was born on 14 Dec 1949 in Miami, Arizona. He was graduated from Miami High School in 1967 and attended Arizona State University from which he received a Bachelor of Science in Engineering degree in June 1971. Upon graduation, he received a commission in the USAF through the ROTC program. He entered active duty in August 1971 as a student in the Communications-Electronics Engineer (3055) course at Keesler Technical Training Center, Keesler AFB, Mississippi. Upon graduation in February 1972 he was assigned to the 1839th Electronics Installation Group, also at Keesler, as Group Electronics Officer and Chief of the Scope CAP (Cable Assessment Program) Branch. Upon acceptance into an exchange program between the Air Force Communications Service and Air Force Systems Command, he was transferred to Headquarters, Space and Missile Systems Organization (SAMSO) at Los Angeles AFS, California in September 1974 where he served as an electrical engineer in the Advanced Ballistic Reentry Systems Program Office. In December 1975 he was transferred to the Global Positioning System Program Office at SAMSO as the antenna development engineer where he served until entering the School of Engineering, Air Force Institute of Technology, in June 1976.

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State-variable feedback is a modern control theory technique that is employed in system design to place closed-loop poles to achieve desired performance characteristics. Two problems associated with state-variable feedback are physical inaccessibility to plant states and output sensitivity to plant parameter variations. In this report an observer is employed to reconstruct all plant states. The plant-observer system is investigated with respect to sensitivity aspects of the following three areas: state-

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variable representation, extent of pole placement, and observer dynamics design. A comparison between physical, phase, and Jordan canonical variables indicated physical and phase variable representations yield identical sensitivity functions; Jordan canonical variables result in greater sensitivity than physical and phase variables. Placing plant poles close together creates a high system sensitivity to plant parameter variations; separation of poles reduces sensitivity. As a minimal requirement for low system sensitivity, observer poles should be placed to the left of dominant plant poles; further reductions in sensitivity are achieved by placing observer poles to the left of all plant poles.